A note on the convex infimum convolution inequality

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Abstract

We characterize the symmetric measures which satisfy the one dimensional convex infimum convolution inequality of Maurey. For these measures the tensorization argument yields the two level Talagrand's concentration inequalities for their products and convex sets in $\mathbb{R}^n$.

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1 Introduction

In the past few decades a lot of attention has been devoted to study the concentration of measure phenomenon, especially the concentration properties of product measures on the Euclidean space $\mathbb{R}^n$. Through this note by $|x|_p$ we denote the l_p norm on $\mathbb{R}^n$, namely $|x|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$, and let us take $B_p^n = \{x \in \mathbb{R}^n : \ |x|_p \leq 1\}$. We say that a Borel probability measure μ on $\mathbb{R}^n$ satisfies concentration with a profile $\alpha_\mu(t)$ if for any set $A \subset \mathbb{R}^n$ with $\mu(A) \geq 1/2$ we have $\mu(A + tB_2^n) \geq 1 - \alpha_\mu(t)$, $t \geq 0$, where B_2^n is the Euclidean ball of radius 1, centred at the origin. Equivalently, for any 1-Lipschitz function f on $\mathbb{R}^n$ we have $\mu(\{x : \ |f - \mathrm{Med}_\mu f| > t\}) \leq \alpha_\mu(t)$, $t > 0$, where $\mathrm{Med}_\mu f$ is a median of f. Moreover, with a slight modification of α_μ, the median can be replaced with the mean, see [L, Proposition 1.8].

A usual way to reach concentration is via certain functional inequalities. For example, we say that μ satisfies Poincaré inequality (sometimes also called the spectral gap inequality) with constant C, if for any $f : \mathbb{R}^n \to \mathbb{R}$ which is, say, C^1 smooth, we have

$$\mathrm{Var}_\mu(f) \leq C \int |\nabla f|^2 \, \mathrm{d}\mu. \tag{1}$$

We assume that C is the best possible constant. This inequality implies the exponential concentration, namely, concentration with a profile $\alpha_\mu(t) = 2\exp(-t/2\sqrt{C})$. It is well known that the product exponential probability measure ν_n, i.e., measure with density $2^{-n}\exp(-|x|_1)$, satisfies (1) with the constant 4 (see Lemma 2.1 in [BL] for a one-line proof of the one dimensional case). From this fact (case $n = 1$) one can deduce, using the transportation argument, that any one-dimensional log-concave measure (i.e. the measure with the density of the form e^{-V}, where V is convex) with variance 1 satisfies the Poincaré inequality with a universal constant C. In fact, there is a so-called Muckenhoupt condition, see [Mu], that fully characterizes measures satisfying the one dimensional Poincaré inequality. Here we state the result due to L. Miclo, see [Mi]. Namely, let us assume, for

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simplicity, that our measure is symmetric. If the density of the absolutely continuous part of μ is equal to p then the optimal constant in (1) satisfies

$$B \le C \le 4B, \qquad \text{where} \qquad B = \sup_{x>0} \left\{ \mu[x,\infty) \int_0^x \frac{1}{p(y)} \, \mathrm{d}y \right\},$$

see [Mi]. In particular, the constant C is finite if $B < \infty$. Suppose that $\mu_1, \ldots, \mu_n$ satisfy Poincaré inequality with constant C. Then the same can be said about the product measure $\mu = \mu_1 \otimes \ldots \otimes \mu_n$, due to tensorization property of (1) (see Corollary 5.7 in [L]), which follows immediately from the subadditivity of the variance. Thus, one can say that the Poincaré inequality is fully understood in the case of product measures.

A much stronger functional inequality that gives concentration is the so-called log-Sobolev inequality,

$$\mathrm{Ent}_\mu(f^2) \le 2C \int |\nabla f|^2 \, \mathrm{d}\mu, \tag{2}$$

where for a function $g \ge 0$ we set

$$\mathrm{Ent}_\mu(g) = \int g \ln g \, \mathrm{d}\mu - \left(\int g \, \mathrm{d}\mu \right) \ln \left(\int g \, \mathrm{d}\mu \right).$$

It implies the Gaussian concentration phenomenon, i.e., concentration with a profile $\alpha_\mu(t) = \exp(-t^2/2C)$. As an example, the standard Gaussian measure γ_n, i.e., the measure with density $(2\pi)^{-n/2} e^{-|x|_2^2/2}$, satisfies (2) with the constant 1 (see [G]). As in (1), the log-Sobolev inequality possesses similar tensorization properties and there is a full description of measures μ satisfying (2) on the real line, see [BG2]. In particular, the optimal constant C in (2) for symmetric μ satisfies

$$\frac{1}{75} B' \le C \le 936 B', \quad \text{where} \quad B' = \sup_{x>0} \left\{ \mu[x,\infty) \ln \left(\frac{1}{\mu[x,\infty)} \right) \int_0^x \frac{1}{p(y)} \, \mathrm{d}y \right\}.$$

These bounds were improved in [BR].

Another way to concentration leads through the so-called property (τ) (see [M] and [LW]). A measure μ on $\mathbb{R}^n$ is said to satisfy property (τ) with a nonnegative cost function φ if the inequality

$$\left(\int e^{f \Box \varphi} \, \mathrm{d}\mu \right) \left(\int e^{-f} \, \mathrm{d}\mu \right) \le 1 \tag{3}$$

holds for every bounded measurable function f on $\mathbb{R}^n$. Here $(f \Box \varphi)(x) = \inf_y \{ f(y) + \varphi(x-y) \}$ is the so-called infimum convolution. Property (τ) implies concentration, see Lemma 4 in [M]. Namely, for every measurable set A we have

$$\mu\left(\{ x \notin A + \{\varphi < t\} \} \right) \le \mu(A)^{-1} e^{-t}.$$

The property (τ) for the standard Gaussian measure γ_n, with the quadratic cost function $\frac{1}{4}|x|_2^2$, can be proved by the use of the so-called Prekopa-Leidler inequality, see [M, Theorem 2]. Based on this idea Bobkov and Ledoux proved a version of property (τ) for log-concave measures $\mathrm{d}\mu = e^{-V} \mathrm{d}x$ with V satisfying strong convexity condition $\lambda V(x) + (1-\lambda) V(y) \ge V(\lambda x + (1-\lambda) y) + \frac{c}{2}(\lambda + o(\lambda)) \|x-y\|^2$, on arbitrary normed space $(\mathbb{R}^n, \|\cdot\|)$, see [BL2].

In the seminal paper [M] Maurey showed that the exponential measure ν_1 satisfies the infimum convolution inequality with the cost function $\varphi(t) = \min\{ \frac{1}{36} t^2, \frac{2}{9}(|t|-2) \}$. As a consequence of

tensorization (see Lemma 1 in [M]) we get that ν_n satisfies (3) with the cost function $\varphi(x) = \sum_{i=1}^n \varphi(x_i)$. This leads to the so-called Talagrand's two level concentration,

$$\nu_n\left(A + 6\sqrt{t}B_2^n + 9tB_1^n\right) \geq 1 - \frac{1}{\nu_n(A)}e^{-t},$$

(see Corollary 1 in [M]). As we can see, this gives a stronger conclusion than the Poincaré inequality. It is a well known fact that the property (τ) with a cost function which is quadratic around 0, say $\varphi(x) = C|x|^2$ for $|x| \leq 1$, implies the Poincaré inequality with the constant $1/(4C)$, see Corollary 3 in [M]. We shall sketch the argument in Section 2. Moreover, Bobkov, Gentil and Ledoux showed that the property (τ) with a cost function $\varphi(x) = \min\{|x|/a, |x|^2/a^2\}$, where $|\cdot|$ is the Euclidean norm on $\mathbb{R}^n$, is equivalent to the Poincaré inequality for smooth $f : \mathbb{R}^n \to \mathbb{R}$ and provided quantitative relation between a and the Poincaré constant, see [BGL].

Clearly, the property (τ) with a cost function φ implies the *weak property* (τ), $\int e^{f\Box\varphi}\,\mathrm{d}\mu \leq e^{\int f\,\mathrm{d}\mu}$. Conversely, the latter implies property (τ) with $\tilde{\varphi}(x) = 2\varphi(x/2)$. It turns out that the weak property (τ) with a cost function φ is equivalent to the so-called transportation-entropy inequality related to φ, namely, the inequality $\inf \mathbb{E}\varphi(X - Y) \leq \int \ln(\mathrm{d}\nu/\mathrm{d}\mu)\mathrm{d}\nu$ valid for all ν, where the infimum runs over all the random vectors (X, Y) whose marginals X, Y have laws μ and ν, respectively. For the proofs of these equivalences see [GL, Section 8.1]. Moreover, there is a full characterization of one-dimensional measures satisfying the transport-entropy inequality (or, equivalently, property (τ)) for a fixed convex cost function φ that is quadratic on some interval $[-h, h]$, $h > 0$, see [Go2].

The goal of this article is to investigate concentration properties of a wider class of measures, i.e., measures that may not even satisfy the Poincaré inequality. For example, in the case of purely atomic measures one can easily construct a non-constant function f with $\int |\nabla f|^2\,\mathrm{d}\mu = 0$. However, one can still hope to get concentration if one restricts A to the class of convex sets. It turns out that to reach exponential concentration for convex sets, it suffices to prove that the measure μ satisfies the convex Poincaré inequality. Below we state the formal definition.

Definition 1. We say that a Borel probability measure μ on $\mathbb{R}$ satisfies the convex Poincaré inequality with a constant C_p if for every convex function $f : \mathbb{R} \to \mathbb{R}$ with f' bounded we have

$$\mathrm{Var}_\mu(f) \leq C_p \int_{\mathbb{R}} (f')^2\,\mathrm{d}\mu. \tag{4}$$

Here we adopt the standard convention that for a locally Lipschitz function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ the gradient f' is defined by

$$f'(x) = \limsup_{h\to 0} \frac{f(x+h) - f(x)}{h}. \tag{5}$$

This definition applies in particular to convex f. If f is differentiable, (5) agrees with the usual derivative.

We also need the following definition.

Definition 2. Let $h > 0$ and $\lambda \in [0, 1)$. Let $\mathcal{M}(h, \lambda)$ be the class of symmetric measures on $\mathbb{R}$, satisfying the condition $\mu[x + h, \infty) \leq \lambda\mu[x, \infty)$ for $x \geq 0$. Moreover, let $\mathcal{M}^+(h, \lambda)$ be the class of measures supported on $\mathbb{R}_+$, satisfying the same condition.

The inequality (4) has been investigated by Bobkov and Götze, see [BG, Theorem 4.2]. In particular, the authors proved (4) in the class $\mathcal{M}(h, \lambda)$ with a constant C_p depending only on h

and λ. This leads to the exponential concentration for 1-Lipschitz convex functions f (as well as the exponential concentration for convex sets) via the standard Herbst argument (see, e.g., Theorem 3.3 in [L]), by deriving the bound on the Laplace transform of f.

It is worth mentioning that a much stronger condition $\mu[x + C/x, \infty) \leq \lambda\mu[x, \infty)$, $x \geq m$, where m is a fixed positive number, has been considered. It implies log-Sobolev inequalities for log-convex functions and the Gaussian concentration of measure for convex sets. We refer to the nice study [A] for the details.

In [M] the author showed that every measure supported on a subset of $\mathbb{R}$ with diameter 1 satisfies the convex property (τ), i.e. the inequality (3) for convex functions f, with the cost function $\frac{1}{4}|x|^2$ (see Theorem 3 therein). The aim of this note is to extend this result. We introduce the following definition.

Definition 3. Define

$$\varphi_0(x) = \begin{cases} \frac{1}{2}x^2 & |x| \leq 1 \\ |x| - \frac{1}{2} & |x| > 1 \end{cases}.$$

We say that a Borel probability measure μ on $\mathbb{R}$ satisfies convex exponential property (τ) with constant C_τ if for any convex function $f : \mathbb{R} \to \mathbb{R}$ with $\inf f > -\infty$ we have

$$\left(\int e^{f\Box\varphi}\, \mathrm{d}\mu\right)\left(\int e^{-f}\, \mathrm{d}\mu\right) \leq 1 \tag{6}$$

with $\varphi(x) = \varphi_0(x/C_\tau)$.

We are ready to state our main result.

Theorem 1. *The following conditions are equivalent*

(a) $\mu \in \mathcal{M}(h, \lambda)$,

(b) There exists $C_p > 0$ *such that* μ *satisfies the convex Poincaré inequality with constant* C_p,

(c) There exists $C_\tau > 0$ *such that* μ *satisfies the convex exponential property* (τ) *with constant* C_τ.

Moreover, (a) implies (c) with the constant $C_\tau = 17h/(1-\lambda)^2$, *(c) implies (b) with the constant* $C_p = \frac{1}{2}C_\tau^2$ *and (b) implies (a) with* $h = \sqrt{8C_p}$ *and* $\lambda = 1/2$.

This generalizes Maurey's theorem due to the fact that any symmetric measure supported on the interval $[-1, 1]$ belongs to $\mathcal{M}(1, 0)$.

It is well known that the convex property (τ) tensorizes, namely, if $\mu_1, \ldots, \mu_n$ have convex property (τ) with cost functions $\varphi_1, \ldots, \varphi_n$ then the product measure $\mu = \mu_1 \otimes \ldots \otimes \mu_n$ has convex property (τ) with $\varphi(x) = \sum_{i=1}^n \varphi_i(x_i)$, see [M, Lemma 5]. Therefore Theorem 1 implies the following Corollary.

Corollary 1. *Let* $\mu_1, \ldots, \mu_n \in \mathcal{M}(h, \lambda)$ *and let us take* $\mu = \mu_1 \otimes \ldots \otimes \mu_n$. *Define the cost function* $\varphi(x) = \sum_{i=1}^n \varphi_0(x_i/C_\tau)$, *where* $C_\tau = 17h/(1-\lambda)^2$. *Then for any convex function we have*

$$\left(\int e^{f\Box\varphi}\, d\mu\right)\left(\int e^{-f}\, d\mu\right) \leq 1. \tag{7}$$

As a consequence, one can deduce the two-level concentration for convex sets and convex functions in $\mathbb{R}^n$.

Corollary 2. *Let $\mu_1, \ldots, \mu_n \in \mathcal{M}(h, \lambda)$. Let $C_\tau = 17h/(1-\lambda)^2$. Take $\mu = \mu_1 \otimes \ldots \otimes \mu_n$. Then for any convex set A with $\mu(A) > 0$ we have*

$$\mu\left(A + \sqrt{2t}C_\tau B_2^n + 2tC_\tau B_1^n\right) \geq 1 - \frac{e^{-t}}{\mu(A)}.$$

Corollary 3. *Let $\mu_1, \ldots, \mu_n \in \mathcal{M}(h, \lambda)$. Let $C_\tau = 17h/(1-\lambda)^2$. Take $\mu = \mu_1 \otimes \ldots \otimes \mu_n$. Then for any convex function $f : \mathbb{R}^n \to \mathbb{R}$ with*

$$|f(x) - f(y)|_2 \leq a|x-y|_2, \qquad |f(x) - f(y)|_1 \leq b|x-y|_1, \quad x, y \in \mathbb{R}^n,$$

we have

$$\mu\left(\{f > Med_\mu f + C_\tau t\}\right) \leq 2\exp\left(-\frac{1}{8}\min\left\{\frac{t}{b}, \frac{t^2}{a^2}\right\}\right), \qquad t \geq 0, \tag{8}$$

and

$$\mu\left(\{f < Med_\mu f - C_\tau t\}\right) \leq 2\exp\left(-\frac{1}{8}\min\left\{\frac{t}{b}, \frac{t^2}{a^2}\right\}\right), \qquad t \geq 0. \tag{9}$$

Let us mention that very recently Adamczak and Strzelecki established related results in the context of modified log-Sobolev inequalities, see [AS]. For simplicity we state their result in the case of symmetric measures. For $\lambda \in [0,1)$, $\beta \in [0,1]$ and $h, m > 0$ the authors defined the class of measures $\mathcal{M}_{AS}^\beta(h, \lambda, m)$ satisfying the condition $\mu([x + h/x^\beta, \infty)) \leq \lambda\mu([x, \infty))$ for $x \geq m$. Note that $\mathcal{M}_{AS}^0(h, \lambda, 0) = M(h, \lambda)$. They proved that products $\mu = \mu_1 \otimes \ldots \otimes \mu_n$ of such measures satisfy the inequality

$$\mathrm{Ent}_\mu(e^f) \leq C_{AS} \int e^f \left(|\nabla f|^2 \vee |\nabla f|^{\frac{\beta+1}{\beta}}_{\frac{\beta+1}{\beta}}\right) \mathrm{d}\mu. \tag{10}$$

for any smooth convex function $f : \mathbb{R} \to \mathbb{R}$. As a consequence, for any convex set A in $\mathbb{R}^n$ with $\mu(A) \geq 1/2$ we have

$$\mu\left(A + t^{\frac{1}{2}} B_2^n + t^{\frac{1}{1+\beta}} C_\tau B_{1+\beta}^n\right) \geq 1 - e^{-C'_{AS} t}, \quad t \geq 0.$$

Here the constants C_{AS}, C'_{AS} depend only on the parameters β, m, h and λ. They also established inequality similar to (8), namely for a convex function f with

$$|f(x) - f(y)|_2 \leq a|x-y|_2, \qquad |f(x) - f(y)|_{1+\beta} \leq b|x-y|_{1+\beta}, \quad x, y \in \mathbb{R}^n,$$

one gets

$$\mu\left(\{f > \mathrm{Med}_\mu f + 2t\}\right) \leq 2\exp\left(-\frac{3}{16}\min\left\{\frac{t^{1+\beta}}{b^{1+\beta}C_{AS}^\beta}, \frac{t^2}{a^2 C_{AS}}\right\}\right), \qquad t \geq 0.$$

However, the authors were not able to get (9). In fact one can show that for $\beta = 0$ our Theorem 1 is stronger than (10). In particular, the inequality (10) is equivalent to

$$\int e^{f \Box \varphi} \, \mathrm{d}\mu \leq e^{\int f \, \mathrm{d}\mu},$$

see [AS], which easily follows from (7).

While writing this note, we were aware of a work in progress by Gozlan, Roberto, Samson, Shu and Tetali (private communication), which shows an equivalence between the convex property (τ) on the real line and certain mass transportation inequalities (see also [GRST]).

The rest of this article is organized as follows. In the next section we prove Theorem 1. In Section 3 we deduce Corollaries 2 and 3.

2 Proof of Theorem 1

We need the following lemma, which is essentially included in Theorem 4.2 of [BG]. For reader's convenience we provide a straightforward proof of this fact.

Lemma 1. *Let $\mu \in \mathcal{M}^+(h,\lambda)$ and let $g : \mathbb{R} \to [0,\infty)$ be non-decreasing with $g(0) = 0$. Then*

$$\int g^2 \,\mathrm{d}\mu \le \left(\frac{2}{1-\lambda}\right)^2 \int (g(x) - g(x-h))^2 \,\mathrm{d}\mu(x).$$

Proof. We first prove that

$$\lambda \int g(x)\,\mathrm{d}\mu(x) \ge \int g(x-h)\,\mathrm{d}\mu(x)$$

for any non-decreasing $g : \mathbb{R} \to [0,\infty)$ such that $g(0) = 0$. Both sides of this inequality are linear in g. Therefore, it is enough to consider only functions of the form $g(x) = \mathbf{1}_{[a,\infty)}(x)$ for $a \ge 0$, since g can be expressed as a mixture of these functions. For $g(x) = \mathbf{1}_{[a,\infty)}(x)$ the above inequality reduces to $\lambda\mu[a,\infty) \ge \mu[a+h,\infty)$, which is clearly true due to our assumption on μ.

The above is equivalent to

$$(1-\lambda)\int g(x)\,\mathrm{d}\mu(x) \le \int (g(x) - g(x-h))\,\mathrm{d}\mu(x). \tag{11}$$

Now, let us use (11) with g^2 instead of g. Then,

$$\begin{aligned}
\int g^2\,\mathrm{d}\mu &\le \frac{1}{1-\lambda}\int (g(x)^2 - g(x-h)^2)\,\mathrm{d}\mu(x) \\
&= \frac{1}{1-\lambda}\int (g(x) - g(x-h))(g(x) + g(x-h))\,\mathrm{d}\mu(x) \\
&\le \frac{1}{1-\lambda}\left(\int (g(x) - g(x-h))^2\,\mathrm{d}\mu(x)\right)^{1/2}\left(\int (g(x) + g(x-h))^2\,\mathrm{d}\mu(x)\right)^{1/2} \\
&\le \frac{2}{1-\lambda}\left(\int (g(x) - g(x-h))^2\,\mathrm{d}\mu(x)\right)^{1/2}\left(\int g(x)^2\,\mathrm{d}\mu(x)\right)^{1/2}
\end{aligned}$$

We arrive at

$$\left(\int g^2\,\mathrm{d}\mu\right)^{1/2} \le \frac{2}{1-\lambda}\left(\int (g(x) - g(x-h))^2\,\mathrm{d}\mu(x)\right)^{1/2}.$$

Our assertion follows. □

In the rest of this note, we take $f : \mathbb{R} \to \mathbb{R}$ to be convex. Let x_0 be a point where f attains its minimal value. Note that this point may not be unique. However, one can check that what follows does not depend on the choice of x_0. Moreover, if f is increasing (decreasing) we adopt the notation $x_0 = -\infty$ ($x_0 = \infty$). Let us define a discrete version of gradient of f,

$$(Df)(x) = \begin{cases} f(x) - f(x-h) & x > x_0 + h \\ f(x) - f(x_0) & x \in [x_0 - h, x_0 + h] \\ f(x) - f(x+h) & x < x_0 - h \end{cases}.$$

Lemma 2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function with $f(0) = 0$ and let $\mu \in \mathcal{M}(h, \lambda)$. Then*

$$\int \left(e^{f/2} - e^{-f/2}\right)^2 \mathrm{d}\mu \leq \frac{8}{(1-\lambda)^2} \int e^{f(x)}((Df)(x))^2 \,\mathrm{d}\mu(x).$$

Proof. Step 1. We first assume that f is non-negative and non-decreasing. It follows that $f(x) = 0$ for $x \leq 0$. Correspondingly, the function $g = e^{f/2} - e^{-f/2}$ is non-negative, non-decreasing and $g(0) = 0$. Moreover, let μ^+ be the normalized restriction of μ to $\mathbb{R}_+$. Note that $\mu^+ \in \mathcal{M}^+(h, \lambda)$. From Lemma 1 we get

$$\begin{aligned}\int \left(e^{f/2} - e^{-f/2}\right)^2 \mathrm{d}\mu &= \frac{1}{2} \int g^2 \,\mathrm{d}\mu^+ \leq \frac{2}{(1-\lambda)^2} \int (g(x) - g(x-h))^2 \,\mathrm{d}\mu^+(x) \\ &= \frac{4}{(1-\lambda)^2} \int (g(x) - g(x-h))^2 \,\mathrm{d}\mu(x).\end{aligned}$$

Observe that

$$\begin{aligned}g(x) - g(x-h) &= e^{\frac{f(x)}{2}} - e^{-\frac{f(x)}{2}} - e^{\frac{f(x-h)}{2}} + e^{-\frac{f(x-h)}{2}} \\ &= \left(e^{\frac{f(x)}{2}} - e^{\frac{f(x-h)}{2}}\right)\left(1 + e^{-\frac{f(x)}{2} - \frac{f(x-h)}{2}}\right) \\ &\leq 2\left(e^{\frac{f(x)}{2}} - e^{\frac{f(x-h)}{2}}\right) \leq e^{\frac{f(x)}{2}}(f(x) - f(x-h)),\end{aligned}$$

where the last inequality follows from the mean value theorem. We arrive at

$$\int \left(e^{f/2} - e^{-f/2}\right)^2 \mathrm{d}\mu \leq \left(\frac{2}{1-\lambda}\right)^2 \int e^{f(x)}((Df)(x))^2 \,\mathrm{d}\mu(x).$$

Step 2. Now let f be non-decreasing but not necessarily non-negative. From convexity of f and the fact that $f(0) = 0$ we get $|f(-x)| \leq f(x)$ for $x \geq 0$. This implies the inequality $|e^{f(-x)} - e^{-f(-x)}| \leq |e^{f(x)} - e^{-f(x)}|$, $x \geq 0$. From the symmetry of μ one gets

$$\int \left(e^{f/2} - e^{-f/2}\right)^2 \mathrm{d}\mu \leq \int \left(e^{f/2} - e^{-f/2}\right)^2 \mathrm{d}\mu^+.$$

Let $\tilde{f} = f\mathbf{1}_{[0,\infty)}$. From Step 1 one gets

$$\begin{aligned}\int \left(e^{f/2} - e^{-f/2}\right)^2 \mathrm{d}\mu^+ &= 2\int \left(e^{\tilde{f}/2} - e^{-\tilde{f}/2}\right)^2 \mathrm{d}\mu \\ &\leq \frac{8}{(1-\lambda)^2} \int e^{\tilde{f}(x)}((D\tilde{f})(x))^2 \,\mathrm{d}\mu(x) \\ &\leq \frac{8}{(1-\lambda)^2} \int e^{f(x)}((Df)(x))^2 \,\mathrm{d}\mu(x).\end{aligned}$$

Step 3. The conclusion of Step 2 is also true in the case of non-increasing functions with $f(0) = 0$. This is due to the invariance of our assertion under the symmetry $x \to -x$, which is an easy consequence of the symmetry of μ and the fact that for $F(x) = f(-x)$ we have $(DF)(x) = (Df)(-x)$.

Step 4. Let us now eliminate of the assumption of monotonicity of f. Suppose that f is not monotone. Then f has a (not necessarily unique) minimum attained at some point $x_0 \in \mathbb{R}$. Due

to the remark of Step 3 we can assume that $x_0 \le 0$. Since $f(0) = 0$, we have $f(x_0) \le 0$. Take $y_0 = \inf\{y \in \mathbb{R} : \ f(y) = 0\}$. Clearly $y_0 \le x_0$. We define

$$f_1(x) = \begin{cases} f(x) & x \ge x_0 \\ f(x_0) & x < x_0 \end{cases}, \qquad f_2(x) = \begin{cases} 0 & x \ge y_0 \\ f(x) & x < y_0 \end{cases}.$$

Note that f_1 is non-decreasing and f_2 is non-increasing. Moreover, $f_1(0) = f_2(0) = 0$. Therefore, from the previous steps applied for f_1 and f_2 we get

$$(e^{f(x_0)/2} - e^{-f(x_0)/2})^2 \mu((-\infty, x_0]) + \int_{x_0}^{+\infty} \left(e^{f/2} - e^{-f/2}\right)^2 \mathrm{d}\mu \le \frac{8}{(1-\lambda)^2} \int_{x_0}^{+\infty} e^{f(x)}((Df)(x))^2 \, \mathrm{d}\mu(x) \tag{12}$$

and

$$\int_{-\infty}^{y_0} \left(e^{f/2} - e^{-f/2}\right)^2 \mathrm{d}\mu \le \frac{8}{(1-\lambda)^2} \int_{-\infty}^{y_0} e^{f(x)}((Df)(x))^2 \, \mathrm{d}\mu(x). \tag{13}$$

Moreover, since $|f(x)| \le |f(x_0)|$ on $[y_0, x_0]$, we have

$$\int_{y_0}^{x_0} \left(e^{f/2} - e^{-f/2}\right)^2 \mathrm{d}\mu \le (e^{f(x_0)/2} - e^{-f(x_0)/2})^2 \mu([y_0, x_0]) \le (e^{f(x_0)/2} - e^{-f(x_0)/2})^2 \mu((-\infty, x_0]). \tag{14}$$

Combining (12), (13) and (14), we arrive at

$$\int \left(e^{f/2} - e^{-f/2}\right)^2 \mathrm{d}\mu \le \frac{8}{(1-\lambda)^2} \int e^{f(x)}((Df)(x))^2 \, \mathrm{d}\mu(x).$$

□

The following lemma provides an estimate on the infimum convolution.

Lemma 3. *Let $C_1, h > 0$. Define $\varphi_1(x) = \frac{1}{C_1}\varphi_0(x/h)$. Assume that a convex function f satisfies $|f'| \le 1/(C_1 h)$. Then*

$$(f \Box \varphi_1)(x) \le f(x) - \frac{C_1}{2}|(Df)(x)|^2.$$

Proof. Let us consider the case when $x \ge x_0 + h$. We take $\theta \in [0, 1]$ and write $y = \theta(x-h) + (1-\theta)x$. Note that $x - y = h\theta$. By the convexity of f we have

$$\begin{aligned}(f \Box \varphi_1)(x) &\le f(y) + \varphi_1(x-y) \le \theta f(x-h) + (1-\theta) f(x) + \varphi_1(h\theta) \\ &= \theta f(x-h) + (1-\theta) f(x) + \frac{1}{2C_1}\theta^2.\end{aligned}$$

Let us now take $\theta = C_1(f(x) - f(x-h))$. Note that $0 \le f' \le 1/C_1 h$ yields $\theta \in [0, 1]$. We get

$$(f \Box \varphi_1)(x) \le f(x) - \theta(f(x) - f(x-h)) + \frac{1}{2C_1}\theta^2 = f(x) - \frac{C_1}{2}(f(x) - f(x-h))^2.$$

The case $x \le x_0 - h$ follows by similar computation (one has to take $y = \theta(x+h) + (1-\theta)x$). Also, in the case $x \in [x_0 - h, x_0 + h]$ it is enough to take $y = \theta x_0 + (1-\theta)x$ and use the fact that $|x - y| = |\theta(x - x_0)| \le h\theta$.

□

Proof of Theorem 1. We begin by showing that (a) implies (c). We do this in three steps.

Step 1. We first show that it is enough to consider only the case when f satisfies $|f'| \le 1/C_\tau$. Let us show that (6) is actually satisfied for all convex $f : \mathbb{R} \to \mathbb{R}$ bounded from below. To this end let us fix f and consider $g(x) = \sup_{y\in\mathbb{R}}\{(f\Box\varphi)(y) - \varphi(x-y)\}$. Since $g(x) = \sup_{y\in\mathbb{R}} \inf_{z\in\mathbb{R}}\{f(z) + \varphi(y-z) - \varphi(x-y)\}$, we get, by taking $z = x$, that $g(x) \le f(x)$ for all $x \in \mathbb{R}$. Moreover, since f is bounded from below, it is easy to see that g is also bounded from below. Since f is convex, one can check that the function $f\Box\varphi$ is also convex (classical property of infimum convolution). Thus, writing $g(x) = \sup_{u\in\mathbb{R}}\{(f\Box\varphi)(x-u) - \varphi(u)\}$, the function g turns out to be convex as a supremum of convex functions. Finally, since $x \mapsto \varphi(x-y)$ is $(1/C_\tau)$-Lipschitz for every y, the function g is also $(1/C_\tau)$-Lipschitz. We check that $g\Box\varphi = f\Box\varphi$. Indeed, the inequality $g\Box\varphi \le f\Box\varphi$ follows from $g \le f$. The other direction is obtained by writing that

$$(g\Box\varphi)(x) = \inf_{y\in\mathbb{R}} \sup_{z\in\mathbb{R}} \inf_{w\in\mathbb{R}} \{f(w) + \varphi(z-w) - \varphi(y-z) + \varphi(x-y)\}$$

and by taking $z = x$. Using $g\Box\varphi = f\Box\varphi$, $g \le f$ and the fact that g is $(1/C_\tau)$-Lipschitz, we arrive at

$$\left(\int e^{f\Box\varphi}\,\mathrm{d}\mu\right)\left(\int e^{-f}\,\mathrm{d}\mu\right) \le \left(\int e^{g\Box\varphi}\,\mathrm{d}\mu\right)\left(\int e^{-g}\,\mathrm{d}\mu\right) \le 1.$$

Step 2. The inequality (6) stays invariant when we add a constant to the function f. Thus, we may and will assume that $f(0) = 0$. Note that from the elementary inequality $4ab \le (a+b)^2$ we have

$$4\left(\int e^{f\Box\varphi}\,\mathrm{d}\mu\right)\left(\int e^{-f}\,\mathrm{d}\mu\right) \le \left(\int \left(e^{f\Box\varphi} + e^{-f}\right)\mathrm{d}\mu\right)^2.$$

Thus, it is enough to show that

$$\int \left(e^{f\Box\varphi} + e^{-f}\right)\mathrm{d}\mu \le 2.$$

Step 3. Take $C_1 = 17/(1-\lambda)^2$, $C_\tau = C_1 h$ and $\varphi(x) = \varphi_0(x/C_\tau)$. Assume that $|f'| \le 1/C_\tau$. By the convexity of φ_0 we get $\varphi(x) \le \frac{1}{C_1}\varphi_0(x/h)$, since $C_1 > 1$. Thus, by Lemma 3 we get $f\Box\varphi \le f(x) - \frac{1}{2}C_1|(Df)(x)|^2$. By the mean value theorem $|(Df)(x)|/h \le 1/C_\tau$. Therefore, $\frac{1}{2}C_1|(Df)(x)|^2 \le \frac{1}{2}C_1(\frac{h}{C_\tau})^2 = 1/2C_1$. Let $\alpha(C_1) = 2C_1(1 - \exp(-\frac{1}{2C_1}))$. The convexity of the exponential function yields $e^{-s} \le 1 - \alpha(C_1)s$, $s \in [0, 1/2C_1]$. Therefore,

$$\begin{aligned}\int \left(e^{f\Box\varphi} + e^{-f}\right)\mathrm{d}\mu &\le \int \left(e^{f(x) - \frac{1}{2}C_1|(Df)(x)|^2} + e^{-f(x)}\right)\mathrm{d}\mu(x)\\ &\le \int \left(e^{f(x)}\left(1 - \frac{1}{2}C_1\alpha(C_1)|(Df)(x)|^2\right) + e^{-f(x)}\right)\mathrm{d}\mu(x).\end{aligned}$$

Therefore, since $e^f + e^{-f} - 2 = (e^{f/2} - e^{-f/2})^2$, we are to prove that

$$\int \left(e^{f/2} - e^{-f/2}\right)^2 \mathrm{d}\mu \le \frac{C_1}{2}\alpha(C_1)\int e^{f(x)}|(Df)(x)|^2\,\mathrm{d}\mu(x).$$

From Lemma 2 this inequality is true whenever $\frac{1}{2}C_1\alpha(C_1) \ge \frac{8}{(1-\lambda)^2}$. It suffices to observe that

$$\frac{1}{2}C_1\alpha(C_1) = C_1^2\left(1 - e^{-\frac{1}{2C_1}}\right) \ge C_1^2\left(1 - \frac{1}{1 + \frac{1}{2C_1}}\right) = \frac{C_1}{2 + \frac{1}{C_1}} \ge \frac{C_1}{2 + \frac{1}{8}} = \frac{8}{(1-\lambda)^2}.$$

We now sketch the proof of the fact that (c) implies (b). Due to the standard approximation argument one can assume that f is a convex C^2 smooth function with bounded first and second derivative (note that in the definition of the convex Poincaré inequality we assumed that f' is bounded). Consider the function $f_\varepsilon = \varepsilon f$. The infimum of $\psi_x(y) = \varphi(y) + \varepsilon f(x-y)$ is attained at the point y satisfying the equation $\psi_x'(y) = \varphi'(y) - \varepsilon f'(x-y) = 0$. Note that ψ_x' is strictly increasing on the interval $[-C_\tau, C_\tau]$. If ε is sufficiently small, it follows that the above equation has a unique solution y_x and that $y_x \in [-C_\tau, C_\tau]$. Thus, $y_x = C_\tau^2 \varepsilon f'(x - y_x)$. This implies $y_x = \varepsilon C_\tau^2 f'(x) + o(\varepsilon)$. We get

$$\begin{aligned} f\Box\varphi(x) = \varphi(y_x) + \varepsilon f(x - y_x) &= \frac{1}{2C_\tau^2} y_x^2 + \varepsilon f(x - \varepsilon C_\tau^2 f'(x)) + o(\varepsilon^2) \\ &= \frac{1}{2}\varepsilon^2 C_\tau^2 f'(x)^2 + \varepsilon f(x) - \varepsilon^2 C_\tau^2 f'(x)^2 + o(\varepsilon^2) \\ &= \varepsilon f(x) - \frac{1}{2}\varepsilon^2 C_\tau^2 f'(x)^2 + o(\varepsilon^2). \end{aligned}$$

Therefore, from the infimum convolution inequality we get

$$\left(\int e^{\varepsilon f(x) - \frac{1}{2}\varepsilon^2 C_\tau^2 f'(x)^2 + o(\varepsilon^2)} \, \mathrm{d}\mu(x)\right)\left(\int e^{-\varepsilon f} \, \mathrm{d}\mu\right) \le 1.$$

Testing (6) with $f(x) = |x|/C_\tau$ one gets that $\int e^{\varphi} \, \mathrm{d}\mu < \infty$ and therefore $\int e^{|x|/C_\tau} \, \mathrm{d}\mu < \infty$. Also, there exists a constant $c > 0$ such that $|f(x)| \le c(1 + |x|)$, $x \in \mathbb{R}$. As a consequence, after some additional technical steps, one can consider the Taylor expansion of the above quantities in $\varepsilon = 0$. This gives

$$\left(\int \left(1 + \varepsilon f - \frac{1}{2}\varepsilon^2 C_\tau^2 f'^2 + \frac{1}{2}\varepsilon^2 f^2 + o(\varepsilon^2)\right) \mathrm{d}\mu\right)\left(\int \left(1 - \varepsilon f + \frac{1}{2}\varepsilon^2 f^2 + o(\varepsilon^2)\right) \mathrm{d}\mu\right) \le 1.$$

Comparing the terms in front of ε^2 leads to

$$\int f^2 \, \mathrm{d}\mu - \left(\int f \, \mathrm{d}\mu\right)^2 \le \frac{1}{2} C_\tau^2 \int f'^2 \, \mathrm{d}\mu.$$

This is exactly the Poincaré inequality with constant $\frac{1}{2}C_\tau^2$.

We show that (b) implies (a). Suppose that a symmetric Borel probability measure μ on $\mathbb{R}$ satisfies the convex Poincaré inequality with a constant C_p. Consider the function $f_u(x) = \max\{x - u, 0\}$, $u \ge 0$. We have

$$\int_{\mathbb{R}} |f_u'(x)|^2 \, \mathrm{d}\mu(x) = \mu([u, \infty)).$$

Since $f_u(y) = 0$ for $y \le 0$ and $\mu((-\infty, 0]) \ge 1/2$, one gets

$$\begin{aligned} \mathrm{Var}_\mu f_u &= \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} (f_u(x) - f_u(y))^2 \, \mathrm{d}\mu\, \mu(x) \, \mathrm{d}\mu\, \mu(y) \ge \frac{1}{2}\int_{\mathbb{R}}\int_{-\infty}^{0} (f_u(x) - f_u(y))^2 \, \mathrm{d}\mu\, \mu(x) \, \mathrm{d}\mu\, \mu(y) \\ &\ge \frac{1}{4}\int_{\mathbb{R}} (f_u(x))^2 \, \mathrm{d}\mu\, \mu(x) \ge \frac{1}{4}\int_{u+\sqrt{8C_p}}^{\infty} (f_u(x))^2 \, \mathrm{d}\mu\, \mu(x) \ge 2C_p \mu\left(\left[u + \sqrt{8C_p}, \infty\right)\right). \end{aligned}$$

These two observations, together with Poincaré inequality, yield that $\mu \in \mathcal{M}(\sqrt{8C_p}, 1/2)$. $\square$

3 Concentration properties

We show that the convex property (τ) implies concentration for convex sets.

Proposition 1. *Suppose that a Borel probability measure μ on $\mathbb{R}^n$ satisfies the convex property (τ) with a non-negative cost function φ, restricted to the family of convex functions. Let $B_\varphi(t) = \{x \in \mathbb{R}^n : \ \varphi(x) \le t\}$. Then for any convex set A we have*

$$\mu(A + B_\varphi(t)) \ge 1 - \frac{e^{-t}}{\mu(A)}.$$

The proof of this proposition is similar to the proof of Proposition 2.4 in [LW]. We recall the argument.

Proof. Let $f = 0$ on A and $f = \infty$ outside of A. Note that f is convex (to avoid working with functions having values $+\infty$ one can consider a family of convex functions $f_n = n\,\mathrm{dist}(A, x)$ and take $n \to \infty$). Suppose that $(f\Box\varphi)(x) \le t$. Then there exists $y \in \mathbb{R}^n$ such that $f(y)+\varphi(x-y) \le t$. Thus, $y \in A$ and $x - y \in B_\varphi(t)$. Therefore $x \in A + B_\varphi(t)$. It follows that $x \notin A + B_\varphi(t)$ implies $(f\Box\varphi)(x) > t$. Applying the infimum convolution inequality we get

$$e^t(1 - \mu(A + B_\varphi(t))) \cdot \mu(A) \le \left(\int e^{f\Box\varphi}\,\mathrm{d}\mu\right)\left(\int e^{-f}\,\mathrm{d}\mu\right) \le 1.$$

Our assertion follows. □

We are ready to derive the two-level concentration for convex sets.

Proof of Corollary 2. The argument is similar to [M, Corollary 1]. Due to Corollary 1, $\mu = \mu_1 \otimes \ldots \otimes \mu_n$ satisfies property (τ) with the cost function $\varphi(x) = \sum_{i=1}^n \varphi_0(x_i/C_\tau)$. Suppose that $\varphi(x) \le t$. Define $y, z \in \mathbb{R}^n$ in the following way. Take $y_i = x_i$ if $|x_i| \le C_\tau$ and $y_i = 0$ otherwise. Take $z_i = x_i$ if $|x_i| > C_\tau$ and $z_i = 0$ otherwise. Then $x = y + z$. Moreover,

$$\sum_{i=1}^n \varphi(y_i/C_\tau) + \sum_{i=1}^n \varphi(z_i/C_\tau) = \sum_{i=1}^n \varphi(x_i/C_\tau) \le t.$$

In particular $|y|_2^2 \le 2C_\tau^2 t$ and $t \ge \sum_{i=1}^n \varphi(z_i/C_\tau) \ge \frac{1}{2}|z|_1/C_\tau$, since $|z_i| - \frac{1}{2} \ge \frac{1}{2}|z_i|$ for $|z_i| \ge 1$. This gives $x \in \sqrt{2t}C_\tau B_2^n + 2tC_\tau B_1^n$. Our assertion follows from Proposition 1. □

Finally, we prove concentration for convex Lipschitz functions.

Proof of Corollary 3. The proof of (8) is similar to the proof of Propositon 4.18 in [L]. Let us define a convex set $A = \{f \le \mathrm{Med}_\mu f\}$ and observe that $\mu(A) \ge 1/2$. Moreover,

$$A + C_\tau(\sqrt{2t}B_2^n + 2tB_1^n) \subset \{f \le \mathrm{Med}_\mu f + C_\tau(a\sqrt{2t} + 2bt)\}.$$

Applying Corollary 2 we get

$$\mu\left(\left\{f > \mathrm{Med}_\mu f + C_\tau(a\sqrt{2t} + 2bt)\right\}\right) \le 2e^{-t}, \qquad t \ge 0.$$

Take $s = C_\tau(a\sqrt{2t} + 2bt)$ and $r = s/C_\tau$. Suppose that $\frac{r}{b} \le \frac{r^2}{a^2}$. Then

$$a\sqrt{2t} + 2tb = r \ge \frac{1}{2}\sqrt{a^2 r/b} + \frac{1}{2}r = a\sqrt{2\frac{r}{8b}} + 2b\frac{r}{4b} \ge a\sqrt{2\frac{r}{8b}} + 2b\frac{r}{8b}.$$

By the monotonicity of $x \mapsto a\sqrt{2x} + 2xb$, $x \geq 0$ it follows that $\frac{1}{8}\min\{\frac{r}{b}, \frac{r^2}{a^2}\} = \frac{r}{8b} \leq t$. Moreover, if $\frac{r}{b} \geq \frac{r^2}{a^2}$, then

$$a\sqrt{2t} + 2tb = r \geq \frac{1}{2}r + \frac{br^2}{2a^2} = a\sqrt{2\frac{r^2}{8a^2}} + 2b\frac{r^2}{4a^2} \geq a\sqrt{2\frac{r^2}{8a^2}} + 2b\frac{r^2}{8a^2}.$$

Therefore, $\frac{1}{8}\min\{\frac{r}{b}, \frac{r^2}{a^2}\} = \frac{r^2}{8a^2} \leq t$. Thus,

$$\mu\left(\{f > \mathrm{Med}_\mu f + rC_\tau\}\right) \leq 2e^{-t} \leq 2\exp\left(-\frac{1}{8}\min\left\{\frac{r}{b}, \frac{r^2}{a^2}\right\}\right), \qquad t \geq 0.$$

For the proof of (9) we follow [T]. Define a convex set $B = \{f < \mathrm{Med}_\mu f - C_\tau(a\sqrt{2t} + 2bt)\}$ with $t \geq 0$. It follows that

$$A + C_\tau(\sqrt{2t}B_2^n + 2tB_1^n) \subset \{f < \mathrm{Med}_\mu f\}$$

and thus Corollary 2 yields

$$\frac{1}{2} \geq \mu\left(A + C_\tau(\sqrt{2t}B_2^n + 2tB_1^n)\right) \geq 1 - \frac{e^{-t}}{\mu(B)}.$$

Therefore $\mu(B) \leq 2e^{-t}$. To finish the proof we proceed as above. □

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